Problem Set II: Due Thursday, April 30 2015

1.) a.) Derive reduced MHD by two different methods. Explain the physics.

b.) What linear waves does reduced MHD support? What happened to the others – i.e. how does the ordering eliminate them? (N.B. It may be useful to read Strauss, '76).

c.) Recover 2D MHD from reduced MHD.

d.) What are the conservation laws of reduced and 2D MHD?

e.) Now, derive the reduced MHD equations when $\underline{B}_o = B_o \hat{z}$ and gravity is present, i.e. $\underline{g} = g\,\hat{x}$.

2.) a.) Derive the Hasegawa-Wakatani equations. Do this like the derivation for reduced MHD, but:

1.) neglect inductive effects, and all magnetic perturbations
2.) retain electron pressure in the Ohm's Law
3.) take electrons isothermal
4.) derive an equation for electron density, including parallel electron flow.

b.) Discuss the conservation properties for this system.

c.) Derive the quasi-linear equations for the H-W system. What do they mean?

d.) Derive the mean vorticity and particle flux.

f.) Relate the vorticity flux to the Reynolds stress.

3.) a.) Starting from the H-W equations, derive the Hasegawa-Mima equation in the limit $k_\parallel^2 v_{The}^2 / \omega \nu \to \infty$. What is the physics of this limit?

b.) What quantities are conserved by the Hasegawa-Mima Equation?

c.) What are the linear waves of the H-M system? Obtain the dispersion relation.

d.) Recover these in the H-W system, for $k_\parallel^2 v_{Th}^2 / \omega \nu > 1$ but not infinite. Discuss your result. How does instability occur?

4.) a.) Show that for incompressible MHD in two dimensions, the basic equations can be written as:

$$(\partial_t + \underline{v} \cdot \nabla)\nabla^2 \phi = (B \cdot \nabla)\nabla^2 A + \nu \nabla^2 \nabla^2 \phi + \tilde{f}$$

$$(\partial_t + \underline{v} \cdot \nabla) A = \eta \nabla^2 A.$$

Here ν is viscosity, η is resistivity, $\underline{v} = \underline{\nabla} \phi \times \hat{z}$ and $\underline{B} = \underline{\nabla} A \times \hat{z}$. $\tilde{f}$ is a random force. Take $P = P(\rho)$.

b.) Take $\underline{B} = B_0 \hat{x}$ to be a weak in-plane magnetic field. Calculate the real frequency and damping for Alfven waves.

c.) Using quasilinear theory, calculate the turbulent resistivity induced by a spectrum of Alfven waves in 2D MHD. For $\nu \to 0$, interpret your result in terms of the freezing-in-law. Why does viscosity enter your result for part i.)? Why does η enter? Contrast these.

d.) Taking $\underline{B} = B_0 \hat{x}$ and $\langle \tilde{V}_y \tilde{A} \rangle = -\eta_T \partial A_0 / \partial y$ as a definition of turbulent resistivity η_T. Show that at stationarity

$$\eta_T = \eta \langle \tilde{B}^2 \rangle / B_0^2,$$

assuming the system has periodic boundary conditions. Discuss your result and its implications. This is a famous result, referred to as the Zeldovich Theorem, after Ya.B. Zeldovich.

e.) What happens if one pair of boundaries are open? (Hint: Consider flux thru surface.)

5.) a.) Derive the tensor virial theorem for a warm, self-gravitating fluid in an external gravitational potential $\phi_{ext}(\underline{x})$. In particular, how does $\phi_{ext}(\underline{x})$ change the virial balance?

b.) Describe the structure of $\phi_{ext}(\underline{x})$, relative to the blob, which is required to confine the fluid.

6.) a.) Kulsrud 5.1

b.) Derive an energy balance theorem for acoustic waves. Discuss the structure.

7.) Kulsrud 5.2 – Ignore the last sentence of the problem.

8.) Kulsrud 5.3

9.) Kulsrud 5.4

10.) Kulsrud 11.1 – Omit the second paragraph.

11.) Kulsrud 7.1

12.) Kulsrud 7.2

13.) Consider a rapidly rotating incompressible fluid. Take $\underline{\Omega} = \Omega_0 \hat{z}$. Obviously, the Coriolis effect is crucial here.

a.) Derive the dispersion relation for inertial waves, with $\omega \sim \Omega$. Take $\nu = 0$. What is the physics of these waves? What is the relation between their group and phase velocity?

b.) Now take $\underline{B}_0 = B_0 \hat{z}$, as well and consider $\omega << \Omega$. For $\nu = \eta = 0$, show that magnetostrophic waves with

$$\omega = \pm \frac{1}{4\pi\rho_0} \left(\underline{k} \cdot \underline{B}_0\right)^2 k / 2(\underline{\Omega} \cdot \underline{k})$$

exist for $(\underline{k} \cdot \Omega)^2 >> (\underline{k} \cdot \underline{B}_0)^2 k^2 / 4\pi\rho_0$. These waves are called magnetostrophic waves.